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HINDU AND GREEK CONTRIBUTIONS TO THE SOLUTION OF THE PELLIAN EQUATION.

BY

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C O N T E N T S

That indeterminate equation of the second degree in two unknowns which is of the form $X^2 - Ay^2 = 1$ has long been called the Pellian equation or the Pellian problem, and it has rarely been mentioned by any other title. John Pell had little to do with it and yet to rename it would be like trying to give another name to North America because Vespucius was not its discoverer. In referring to this as the Pellian equation, modern mathematicians have not disregarded Fermat, Wallis, Brouncker, Gauss and the hundreds of others who have contributed to the history of the subject, just as in permitting the reciprocity theorem to bear the name of Legendre, the world has not failed to recognize the work of Kronecker. Ifthe name were changed, however, then such expressions as Pellian convergents, Pellian terms, Pellian numbers and Pellian factorisations, all depending upon the connection of Pell's name with the equation, would become illogical and mere arbitrary designations.

The name originated in a mistaken notion of Leonard Euler that John Pell was the author of the solution which was really the work of Lord Brouncker. Euler in his cursory reading of Wallis's algebra[1] must have confused the contributions of Pell and Brouncker. Wallis[2] gives Pell credit for certain researches in indeterminate analysis, but where $Ay^2 \; 1 = X^2$ is discussed, only Brouncker's methods are set forth. The assertion of Hankel[3], for example, that Pell treated of the equation $Ay^2 \; 1 = X^2$ rests upon a misunderstanding. Hankel said, "Pell has done it no other service than to set it forth again in a much read work", i.e. in the notes to the English translation which Brancker, in 1668, published of the "Teutschen Algebra" of Rahn. Konen[4] says that, in the only copy which he could obtain of this work, there is nothing relative to this equation and he thinks that it is very probable that Pell never considered it. Eneström[5] holds the same opinion, basing it upon the examination of three copies of Rahn's Algebra. Nevertheless it seems not improbable that Pell solved the equation, for we find it discussed in Rahn's algebra[6] under the form $X = 12yy -$ [illegible]. This shows that Pell had some acquaintance with the Pellian equation and that Euler was not so far out of the way when he attributed to Pell work upon it.

1. G. Wertheim, "Über den Ursprung des Ausdruckes 'Pellsche Gleichung'" Bibliotheca Mathematica, vol. 11 (3), p.360, Leipzig, 1901.

2. J. Wallis, "Opera Mathematica", vol. 11, chap. 98, p.418, Oxford, 1693.

3. H. Hankel, "Zur Geschichte der Mathematik in Alterthum und Mittelalter", p.203, Beipzig, 1874.

4. M. Konen, "Geschichte der Gleichung $t^2-Du^2 =$", Leipzig, 1901.

5. G. Eneström, "Über den Ursprung der Bennenung Pell'sche Gleichung", Bibliotheca Mathematica, vol. 111 (3), p.204, Leipzig, 1902.

6. J. H. Rahn, "An introduction to algebra, translated out of the High Dutch into English by Thomas Brancker, M.A. much altered and augmented by D.P.", p.143, London 1668.

In the Pellian equation and in such generalizations as $X^2-Ay^2=\pm 1$, $X^2-Ay^2=\pm b$, $X^2-Ay^2=\sigma^2$ it has been customary to restrict A to a positive non-square integer and to seek integral solutions for X and y. It is evident that no generality will be lost if the solutions are further restricted to positive integers. If it were permitted to make $A<-1$ there would be only two solutions in integers, $X=\pm 1$, $y=0$. For $A=-1$, only four solutions, $X=\pm 1$, $y=0$; $X=0$, $y=\pm 1$. But for $A>1$, $X^2-Ay^2=1$ has infinitely many solutions. However from every two solutions X_1, y_1 and X_2, y_2 - the same or different - a third can be secured from the following identity, $(X_1+y_1\sqrt{A})(X_2+y_2\sqrt{A}) = X_1X_2+Ay_1y_2+(X_1y_2+X_2y_1)\sqrt{A}$, namely,

$$X = X_1X_2+Ay_1y_2$$

$$y = X_1y_2+X_2y_1$$

X_1 y_1 is the fundamental solution - the solution in smallest integers - neither however being 0.

To illustrate: Given the equation $X^2-3y^2=1$.

Here $X_1=2$ $\quad X_2=7$
$\quad\quad y_1=1$ $\quad y_2=4$ $\quad$ A third solution can be secured

from the identity $(2+\sqrt{3})(7+4\sqrt{3}) = 14+15\sqrt{3}+12 = 26+15\sqrt{3}$, namely, $X=26$ and $y=15$.

It is evident that the Pellian equation is closely connected with the primitive methods of approximating a square root. Among the first definite traces that we have of these methods of approximation are those found in the dimensions of ancient structures, such as the Egyptian pyramids, the Parthenon and other temples on the Acropolis at Athens, and altars and temples in many places. For example, the principal room, called the King's chamber[1], in the pyramid of Cheops, has for the ratio of its height to its breadth about 1.117 or nearly $\frac{1}{2}\sqrt{5}$ showing that the architect must have known the approximate value of this surd and it is suggestive when we note that this is one half the ratio X : y of some of the solutions of $X^2-5y^2=1$. The ratio 17 : 12 is found many times on the Acropolis and $X=17$, $y=12$ satisfies $X^2-2y^2=1$. These solutions are interwoven with the number theory of Pythagoras and his followers and the ancient tradition that Pythagoras had obtained his knowledge of numbers by a sojourn upon the Euphrates is strengthened by recent discoveries[2]. These solutions are found in connection with the mystic Platonic number.[3]

The problem of the extraction of square roots is not attached to the decimal system, as is the present usual custom, nor to the sexagesimal system as was the custom of the Greek astronomers. The fact that tables from the temple library at Nippur and from the library of Sardanapulus IV at Babylon has an important bearing on the methods of solution of the Pellian equation. It is therefore quite possible that solutions of this equation extend back to the ancient Babylonians 4000 years ago.

The first approximation to $\sqrt{2}$ appeared both in India and in Greece about 400 B.C. The younger Pythagoreans (before 410 B.C.) recognized and proved the incommensurability of the diagonal and side of a square and set forth certain approximations. The Sulva-sutras

in India, which contained approximations to $\sqrt{2}$, are not later than the fourth or fifth century before Christ. The fact that these approximations appeared about the same time in Greece and India and that the first step in these approximations appears so simple, indicate the independence of the Hindu and Greek mathematicians. The processes of root approximations were developed differently by the Hindus and the Greeks. This adds probability to the idea of independent discovery. Indeed, we fail to find any definite connection between the Greek and the Hindu algebra. Each starts with a simple approximation to the square root of a number and seeks from this a closer approximation. The Hindus were skilful calculators but mediocre theorists, while the genius of the Greeks tended more towards geometry.

1. H. A. Naber, "Das Theorem des Pythagoras", p.48, Haarlem, 1908.

2. H. V. Hilprecht, "Mathematical, meteorological and chronological tablets from the temple library at Nippur", vol. XX, part 1 of Series A, cuneiform texts, University of Pennsylvania, Phila., 1906.

3. S. Günther, "Die Platonische Zahl", p.5, Dresden, 1882.

One of the monuments of the old Hindu mathematicians is that curious work, the Sulva-sutras, or "Precepts of line". These books contain rules to be observed by the Brahmins in the construction of their altars and give close approximations for $\sqrt{2}$. Baudhayâna, the author of the oldest of these works, uses first the approximation $\frac{17}{12}$.[1] He gives this rule in relation to the approximation for the diagonal of a square, "Increase the measure by its third part and this third part by its own fourth less the thirty-fourth part of that fourth". The rule evidently means that $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} = \frac{577}{408}$. The result $\frac{577}{408}$ furnishes a solution of the Pellian equation $X^2 - 2y^2 = 1$, i.e. $332,929 - 332,928 = 1$.

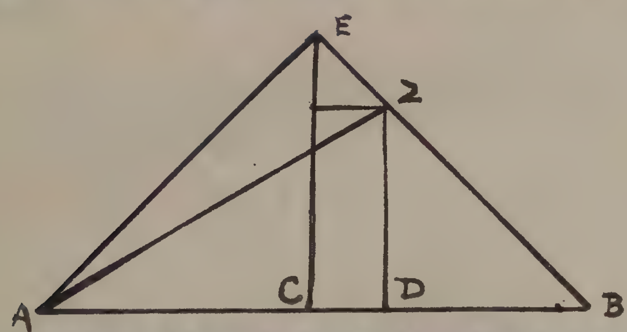
Perhaps some of the first approximations were found by trial with pebbles, taking a square number and attempting to arrange in two equal squares and repeating the process with other squares until close approximations were obtained. If 289 pebbles arranged in a square were taken up and rearranged in two squares of 144 each with 1 excess, a solution of the Pellian equation $X^2 - 2y^2 = 1$ would be found in which $x = 17$, $y = 12$.

Suppose $\frac{p}{q}$ to be a first approximation of the square root of a number A, and that another is built up from it, thus, $\sqrt{A} \sim \frac{p}{q}$, $\sqrt{A} \sim \frac{p}{q} + \frac{R}{2pq} = \frac{p_1}{q_1}$, where $p^2 - Aq^2 = -R$, and $\sim$ is to read "approximates". Let us examine the first extraction of $\sqrt{2}$ of which we have knowledge, those of the Sulva-sutras just mentioned. Suppose $\frac{3}{2}$ has been found by experiment as an approximation, $\sqrt{2} \sim \frac{3}{2}$, we have $\sqrt{2} \sim \frac{3}{2} + \frac{-1}{2.3.2} = \frac{17}{12}$ and $\sqrt{2} \sim \frac{17}{12} + \frac{-1}{2.17.12} = \frac{577}{408}$ where 3,2; 17, 12; 577,408 are solutions of $X^2 - 2y^2 = 1$. Where $R = -1$, we have limitless series of solutions for the Pellian equation $X^2 - Ay^2 = 1$.

Baudhayâna sometimes used for $\sqrt{3}$ the approximation $\frac{26}{15}$; for he gave for the area of a circle $\left(\frac{13d}{15}\right)^2 = \frac{1}{4} d^2 \pi$. Here π is evidently set equal to 3, as in the early writings of all oriental nations. Therefore $3 \sim \frac{13^2}{15^2} \cdot 4$ and $\sqrt{3} \sim \frac{13}{15} \cdot 2 \sim \frac{26}{15}$. Baudhayâna also knew the more exact value, $\sqrt{3} \sim \frac{26}{15}$, $\sqrt{3} \sim \frac{26}{15} + \frac{-1}{2.26.15} = \frac{1351}{780}$. Both 26, 15 and 1351, 780 are solutions of the Pellian equation $X^2 - 3y^2 = 1$.

1. M. Simon, "Geschichte der Mathematik in Altertum", p.157 Berlin, 1909.

With the same starting point, $\sqrt{A} \sim \frac{p}{q}$, the Greeks were led by somewhat different rules to closer approximations of $\sqrt{A}$ but still b methods in which the approximations satisfy the equations $X^2 - Ay^2 = \pm 1$.

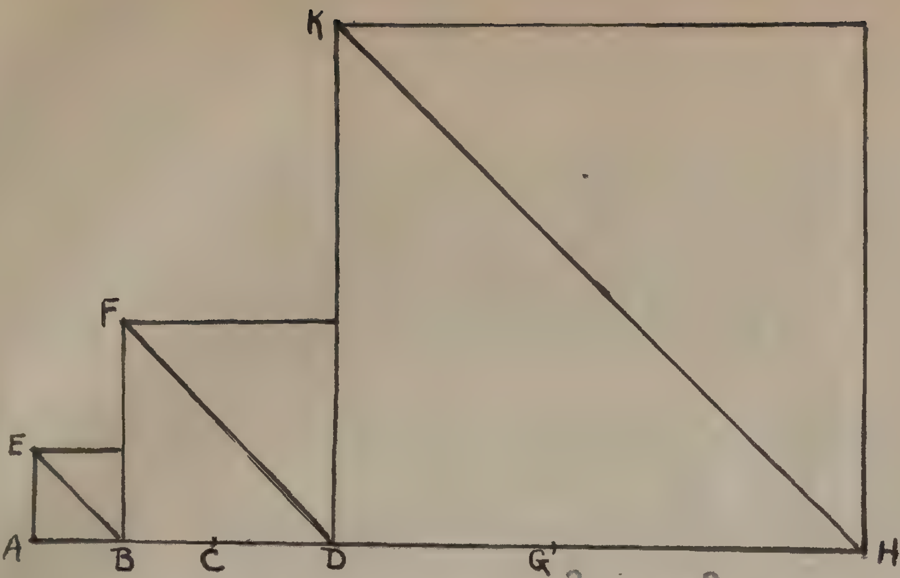


If C (Fig.1) is the mid-point of the base of an isosceles right triangle AEB, and D any point on AB, then $\overline{AD}^2 + \overline{DB}^2 = 2\overline{AC}^2 + 2\overline{CD}^2$. For if Z is the point where the perpendicular at D meets EB, $\overline{DB}^2 = \overline{DZ}^2$ for $\angle B = 45^\circ$ and therefore $\angle DZB = 45^\circ$. Therefore $\overline{AD}^2 + \overline{DB}^2 = \overline{AD}^2 + \overline{DZ}^2 = \overline{AZ}^2 = \overline{AE}^2 + \overline{EZ}^2 = 2\overline{AC}^2 + 2\overline{CD}^2$. Then $2\overline{AC}^2 - \overline{AD}^2 = \overline{DB}^2 - 2\overline{CD}^2$. If $CD = q$ and $BD = p$, $AD = BC + CD = p + q + q = p + 2q = p_1$. $AC = BC = q + p = q_1$. Then $2q_1^2 - p_1^2 = -(2q^2 - p^2)$. This enables us to derive from one integral solution, p, q, of one of the two equations $X^2 - 2y^2 = \pm 1$, a solution, p_1, q_1, for the other, always in larger integers.

To illustrate : $X^2 - 2y^2 = -1$. $p = 1$, $q = 1$ is a solution. $p_1 = p + 2q = 1 + 2 = 3$; $q_1 = q + p = 1 + 1 = 2$. Now $p_1 = 3$; $q_1 = 2$ is a solution of $X^2 - 2y^2 = 1$.

Also $p = 7$, $q = 5$ is a solution of $X^2 - 2y^2 = -1$. $p_1 = p + 2q = 7 + 10 = 17$, $q_1 = q + p = 5 + 7 = 12$. Now $p_1 = 17$; $q_1 = 12$ is a solution of $X^2 - 2y^2 = 1$.

As Proclus pointed out, the Pythagoreans proceeded as follows: On AB (Fig.2) construct a square and draw its diagonal BE, on the prolongation of AB lay off BC = AB and CD = BE. Then $\overline{CD}^2 = 2\overline{AB}^2$, according to the Pythagorean Theorem. $\overline{AD}^2 + \overline{CD}^2 = (AB+BD)^2 + (BD-AB)^2 = 2\overline{AB}^2 + 2\overline{BD}^2$



Consequently, after subtraction, $\overline{AD}^2 = 2\,\overline{BD}^2$. Now erect on BD a square and draw its diagonal. Then $\overline{DF}^2 = 2\,\overline{BD}^2$. Therefore $\overline{DF}^2 = \overline{AD}^2$ and DF = AD = 2 AB + BE. Upon the prolongation of AD, lay off DG = BD, and G H = DF; and on DH erect a square and draw its diagonal HK. Then, by similar steps, HK = BH = 2 BD + DF. It is clear that, in the continuation of this construction, in each new square the diagonal equals the sum of the diagonal and double the side of the previous square.

If we designate the sides of the successive squares by s_1, s_2, s_3, ..., and the corresponding diagonals by d_1, d_2, d_3, ..., we have the double series,

$$
\begin{array}{lll}
s_1 & s_2 = s_1 + d_1 & s_3 = s_2 + d_2 \ldots\ldots \\
d_1 & d_2 = 2s_1 + d_1 & d_3 = 2s_2 + d_2 \ldots\ldots
\end{array}
$$

in which the double square of each term in the upper line is equal to the square of the corresponding term in the lower line.

This geometric proof was, however, for the Pythagoreans the introduction to some fine considerations in the theory of numbers. If they set s = 1, then d = $\sqrt{2}$, an "unspeakable" number and in this connection comes into view the ῥητὴ διάμετρος, the nearest integer, the number 1. Then they compared the square of the ἄρρητος and the ῥητὴ διάμετρος, the first 2, the second 1, the difference 1. Distinguishing the successive ῥηταὶ διάμετροι as $\delta_1, \delta_2, \delta_3, \ldots$ and substituting them in the double series they change to,

$$
\begin{array}{lll}
s_1 = 1 & s_2 = s_1 + \delta_1 = 2 & s_3 = s_2 + \delta_2 = 5 \ldots\ldots \\
\delta_1 = 1 & \delta_2 = 2s_1 + \delta_1 = 3 & \delta_3 = 2s_2 + \delta_2 = 7 \ldots\ldots
\end{array}
$$

Now no longer, as in the first double series, does the square of each lower member equal double the square of the upper, but the difference is constantly 1 and alternately + and −. Each $\delta_n s_n$ is a solution of one of the Pellian equations $\delta^2 - 2s^2 = \pm 1$.

Besides the geometric motive which led to the discovery of approximations for $\sqrt{2}$, problems of music also led to new researches to determine as exactly as possible $\sqrt{2}$. The Pythagorean concept of the harmonic mean, as its name implies, is derived from a problem of music. As the solutions of $X^2 - 2y^2 = \pm 1$ were evidently made in order to obtain approximations to the $\sqrt{2}$, so when we find Archimedes[1] giving without explanation the fractions $\frac{265}{153}$ and $\frac{1351}{780}$ as approximations to $\sqrt{3}$, the most natural hypothesis is that he obtained them from similar equations with 3 substituted for 2. Archimedes was the first Greek mathematician who was not content to speculate over irrationals but handled them in computation. He expressed the sides of regular inscribed and circumscribed polygons in terms of the radius of the circle. In particular for a regular hexagon he gave the proportion $r : \frac{a}{2} = \sqrt{3} : 1$ and then set $\frac{1351}{780} > \sqrt{3} > \frac{265}{153}$. There has been much speculation as to how these approximations were obtained. If $\frac{p}{q}$ is an approximation for the $\sqrt{3}$, then $\frac{3q}{p}$ is and we take $p_1 = p + 3q$, and $q_1 = p + q$. Then $p_1^2 - 3q_1^2 = -2\,(p^2 - 3q^2)$

To illustrate, if $p = 2$ and $q = 1$, the difference $= +1$
$p_1 = 5$, $(p + 3q)$ and $q_1 = 3$, $(p + q)$, the difference $= -2$
$p_2 = 14$, $(p_1 + 3q_1)$ and $q_2 = 8$, $(p_1 + q_1)$, the difference $= 4$, $(196-192)$, etc.

So we see that if we start from the difference $+1$ (as for $p = 2$ and $q = 1$), the following difference will be -2 and the next $+4$. But the terms of the corresponding relations are even numbers, as it is easy to see. Therefore if we take $2p_2 = p_1 + 3q_1$ and $2q_2 = p_1 + q_1$ we find $+1$ again for a difference, thus: $p_2^2 - 3q_2^2 = 1$. To illustrate, p_2 and q_2, instead of being equal to 14 and 8 respectively as above, are taken as 7 and 4 respectively and we then have $7^2 - 3.4^2 = 1$. We obtain then a complete series of solutions as follows; writing the solutions for $X^2 - 3y^2 = 1$ and $X^2 - 3y^2 = -2$ in separate rows,

t row $\frac{2}{1}$, $\frac{14-7}{8-4}$, $\frac{52-26}{30-15}$, $\frac{194-97}{112-56}$, $\frac{724-362}{418-209}$, $\frac{2702-1351}{1560-780}$,

d row $\frac{5}{3}$, $\frac{19}{11}$, $\frac{71}{41}$, $\frac{265}{153}$, $\frac{989}{571}$,

The approximation in the second row is obtained from the approximation slightly to the left in the first row by the use of $p_1 = p + 3q$ and $q_1 = p + q$. So also are the approximations in the first row obtained from those in the second row slightly to the left. Let us see how an approximation in the first row is obtained from the approximation next to it and on the left in the same row.

$$\frac{p}{q} \qquad \frac{p+3q}{p+q} \qquad \frac{p + 3q + 3\,(p + q)}{p + 3q + p + q} = \frac{2p + 3q}{p + 2q}$$

So we see that an approximation either in the first row or second row is obtained from the approximation next to it and on the left and in the same row by the use of $p_1 = 2p + 3q$ and $q_1 = p + 2q$. From

the above rows Archimedes could choose the terms which were convenient for the degree of approximation that he desired.

1. T. L. Heath, "Works of Archimedes", p. LXXX, Cambridge 1897.

Heron of Alexandria is noted for his approximation to $\sqrt{3}$. He used $\frac{26}{15}$ in connection with finding the area of an equilateral triangle. He found approximations to many other surds, giving results which correspond exactly to solutions of the Pellian equation. The form in which Heron expresses the approximations is suggestive of Egyptian methods. The following are a few of his approximations with their connection indicated:

urd		Heron's Approximation	Equivalent fraction	Corresponding to the fundamental solution of
$\sqrt{50}$	$\sim$	$7+\frac{1}{14}$	$\frac{99}{14}$	$x^2 - 50y^2 = 1$
$\sqrt{135}$	$\sim$	$11+\frac{1}{2}+\frac{1}{14}+\frac{1}{21}$	$\frac{244}{21}$	$x^2 - 135y^2 = -1$
$\sqrt{1575}$	$\sim$	$39+\frac{2}{3}+\frac{1}{51}$	$\frac{2024}{51}$	$x^2 - 1575y^2 = 1$
$\sqrt{216}$	$\sim$	$14+\frac{2}{3}+\frac{1}{33}$	$\frac{485}{33}$	$x^2 - 216y^2 = 1$
$\sqrt{720}$	$\sim$	$26+\frac{1}{2}+\frac{1}{3}$	$\frac{161}{6}$	$x^2 - 720y^2 = 1$

Diophantus flourished about 250 A.D. Tannery thinks that iophantus may have discussed the equation $X^2 - Ay^2 = 1$ more fully in at part of the Arithmetica which is lost. This is his suggestion of w Diophantus might have proceeded to find a more general solution, p_1, q_1, from a given solution p, q, of the equation $X^2 - Ay^2 = 1$. Let $p_1 = mx - p$, and $q_1 = x + q$; then $p_1^2 - Aq_1^2 = m^2x^2 - 2mpx + p^2 - Ax^2 - 2Aqx - Aq^2 = 1$. Then since $p^2 - Aq^2 = 1$, $X = \frac{2mp + 2Aq}{m^2 - A}$.

$$p_1 = \frac{(m^2+A)\cdot p + 2\,Amq}{m^2 - A}, \qquad q_1 = \frac{2\,mp + (m^2+A)q}{m^2 - A}$$

d in fact $p_1^2 - Aq_1^2 = 1$. If an integral solution is wanted it n be had by substituting $\frac{u}{v}$ for m where u, v, is a solution of the uation $u^2 - Av^2 = 1$; i.e., another solution of the original equation. en $p_1 = (u^2 + Av^2)\cdot p + 2\,Auvq$, $q_1 = 2\,puv + (u^2 + Av^2)\cdot q$. In order to ve the above values for p_1 q_1 correspond with the simpler values ven on page 6 for the solution of the equation $X^2 - 3y^2 = 1$, it is cessary to take u, v, as a solution of the equation $X^2 - 3y^2 = -2$. en when u = 1, v = 1, satisfy the equation $X^2 - 3y^2 = -2$, we have

$$= \frac{(u^2+3v^2)p + 6\,uvq}{u^2 - 3v^2} = \frac{4p+6q}{-2} = -(2p+3q) \text{ and } q_1 = \frac{2\,uvp + (u^2+3v^2)q}{u^2 - 3v^2} =$$

$\frac{2p+4q}{-2} = -(p+2q)$ and the positive values for p_1, q_1, could also be taken since they satisfy the equation $p_1^2 - 3q_1^2 = 1$. Thus in order to get a general solution, Diophantus required two known solutions of the original equation, or one of the original and one of an auxiliary equation.